MULTIVARIABLE LUBIN-TATE (φ, Γ)-MODULES AND FILTERED φ-MODULES

by

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Abstract. — We define some rings of power series in several variables, that are attached to a Lubin-Tate formal module. We then give some examples of (φ, Γ)-modules over those rings. They are the global sections of some vector bundles on the p-adic open unit polydisk, that are constructed from a filtered φ-module using a modification process. We prove that we obtain every crystalline (φ, Γ)-module over those rings in this way.

Contents

Introduction

Let F be the unramified extension of $\mathbf{Q}_p$ of degree h and let $q = p^h$ so that the residue field of $\mathcal{O}_F$ is $\mathbf{F}_q$. We fix an embedding $F \subset \overline{\mathbf{Q}}_p$ so that if $\sigma : F \to F$ denotes the absolute Frobenius map, which lifts $x \mapsto x^p$ on $\mathbf{F}_q$, then the h embeddings of F into $\overline{\mathbf{Q}}_p$ are given by $\mathrm{Id}, \sigma, \ldots, \sigma^{h-1}$. The symbol φ_q denotes a σ^h-semilinear Frobenius map. If K is a subfield of $\overline{\mathbf{Q}}_p$, then let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$.

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The goal of this short note is to present a first attempt to construct some "multivariable Lubin-Tate (φ,Γ)-modules", that is some (φ_q,Γ_F)-modules over rings of power series in h variables, on which $\Gamma_F = \mathcal{O}_F^\times$ acts by a formula arising from a Lubin-Tate formal $\mathcal{O}_F$-module. A construction of such (φ_q,Γ_F)-modules, but "in one variable", was carried out by Kisin and Ren in [**KR09**]: they prove that in certain cases, the (φ_q,Γ_F)-modules arising from Fontaine's standard construction of [**Fon90**] are overconvergent. In order to do so, Kisin and Ren adapt the constructions of (φ,Γ)-modules attached to filtered (φ,N)-modules given in [**Ber08b**] to their setting, which allows them to attach a (φ_q,Γ_F)-module in one variable to a filtered φ_q-module. They then point out in the introduction of [**KR09**] that "it seems likely that in order to obtain a classification valid for any crystalline G_K-representation one needs to consider higher dimensional subrings of $W(\mathrm{Fr}\ R)$, constructed using the periods of all the conjugates of [the Lubin-Tate group]".

The motivation for these computations is the hope that we can construct some representations of the Borel subgroup of $\mathrm{GL}_2(F)$, for example using the recipe given by Colmez in [**Col10**], that would shed some light on the p-adic local Langlands correspondence for $\mathrm{GL}_2(F)$ (see [**Bre10**]). Theorems A, B and C below are a very first step in this direction, but remain insufficient. In particular, the "p-adic Fourier theory" of Schneider and Teitelbaum (see [**ST01**]) will very likely play an important role in the sequel.

We now describe our results in more detail. Let LT_h be the Lubin-Tate formal $\mathcal{O}_F$-module for which multiplication by p is given by $[p](T) = pT + T^q$. We denote by $[a](T)$ the element of $\mathcal{O}_F[\![T]\!]$ that gives the action of $a \in \mathcal{O}_F$ on LT_h. We consider two rings $\mathcal{R}^+(Y)$ and $\mathcal{R}(Y)$ of power series in the h variables $Y_0,\dots,Y_{h-1}$, with coefficients in F. The ring $\mathcal{R}^+(Y)$ is the ring of power series that converge on the open unit polydisk, and $\mathcal{R}(Y)$ is the Robba ring that corresponds to it, by adapting Schneider's construction given in the appendix of [**Záb12**]. The action of the group $\mathcal{O}_F^\times$ on those rings is given by the formula $a(Y_j) = [\sigma^j(a)](Y_j)$, and the Frobenius map φ_q is given by $\varphi_q(Y_j) = [p](Y_j)$.

The construction of p-adic periods for Lubin-Tate groups gives rise to a map $\mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, where $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is the Fréchet completion of $\widetilde{\mathbf{B}}^+ = W(\widetilde{\mathbf{E}}^+)[1/p]$, and we prove (corollary 3.7) that this map is in fact injective (if $\widetilde{\mathcal{R}}^+(Y)$ denotes the completion of the perfection of $\mathcal{R}^+(Y)$, then the map above extends to a map $\widetilde{\mathcal{R}}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ but note that, by the theory of the field of norms of [**FW79**] and [**Win83**], this latter map is not injective anymore if $h \geqslant 2$. This has prevented us from studying étale φ_q-modules using Kedlaya's methods, so such considerations are absent from this note).

Let D be a finite dimensional F-vector space, endowed with an F-linear Frobenius map $\varphi_q : D \to D$, and an action of G_F on D that factors through Γ_F and commutes with φ_q. For each $0 \leqslant j \leqslant h-1$, let $\mathrm{Fil}_j^\bullet$ be a filtration on D that is stable under Γ_F.

For example, if V is an F-linear crystalline representation of G_F of dimension d, then $\mathrm{D}_{\mathrm{cris}}(V)$ is a free $F \otimes_{\mathbf{Q}_p} F$-module of rank d, and we have $\mathrm{D}_{\mathrm{cris}}(V) = D \oplus \varphi(D) \oplus \cdots \oplus \varphi^{h-1}(D)$, according to the decomposition of $F \otimes_{\mathbf{Q}_p} F$ as $\prod_{\sigma^i : F \to F} F$. Each $\varphi^j(D)$ has the filtration induced from $\mathrm{D}_{\mathrm{cris}}(V)$, and we set $\mathrm{Fil}_j^k D = \varphi^{-j}(\mathrm{Fil}^k \mathrm{D}_{\mathrm{cris}}(V) \cap \varphi^j(D))$.

The composite of the map $\mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ with the map $\varphi^{-k} : \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ gives rise to a map $\iota_k : \mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. Let $\log_{\mathrm{LT}}(T)$ be the logarithm of LT_h, and let $\lambda = \prod_{j=0}^{h-1} \log_{\mathrm{LT}}(Y_j)/Y_j$ (note that the image of $\prod_{j=0}^{h-1} \log_{\mathrm{LT}}(Y_j)$ in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is some multiple of $t = \log(1+X)$, so that λ is an analogue of t/X). Define

$$\mathrm{M}^+(D) = \{y \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D,\ \iota_k(y) \in \mathrm{Fil}^0_{-k}(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^{-k}} D) \text{ for all } k \geqslant h\}.$$

Theorem A. — *The $\mathcal{R}^+(Y)$-module $\mathrm{M}^+(D)$ is free of rank $\dim D$.*

The definition of $\mathrm{M}^+(D)$ is analogous to the one given in [**Ber08b**], [**KR09**] and similar articles. When $h=1$, the proof of theorem A relies on the fact that $\mathrm{M}^+(D)$ can be seen as a vector bundle on the open unit disk. Our proof of theorem A relies on the one dimensional case, and on the interpretation of $\mathrm{M}^+(D)$ as a vector bundle on the open unit polydisk. These vector bundles were studied by Gruson in [**Gru68**] and Bartenwerfer in [**Bar81**].

Let $\mathrm{M}(D) = \mathcal{R}(Y) \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$, so that $\mathrm{M}(D)$ is free of rank $\dim D$ over the Robba ring $\mathcal{R}(Y)$.

Theorem B. — *If D and $\mathrm{M}(D)$ are as above, then:*

1. *$\mathrm{M}(D)$ is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$;*
2. *if V is an F-linear crystalline representation of G_F, and if D arises from $\mathrm{D}_{\mathrm{cris}}(V)$ as above, then there is a natural map $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathcal{R}(Y)} \mathrm{M}(D) \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_F V$, and this map is an isomorphism.*

If M is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, then we define $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = (\mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M})^{\Gamma_F}$ and we say that M is crystalline if $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = \mathrm{rk}(\mathrm{M})$. For example, if D is a filtered φ_q-module with h filtrations $\mathrm{Fil}_j^\bullet$ as above, and on which the action of G_F is trivial, then $\mathrm{M}(D)$ is a crystalline (φ_q, Γ_F)-module.

Theorem C. — *The functors $\mathrm{M} \mapsto \mathrm{D}_{\mathrm{cris}}(\mathrm{M})$ and $D \mapsto \mathrm{M}(D)$, between the category of crystalline (φ_q, Γ_F)-modules over $\mathcal{R}(Y)$ and the category of φ_q-modules with h filtrations, are mutually inverse and give rise to an equivalence of categories.*

We now give a short description of the contents of this note: in §1, we give some reminders about the p-adic periods of Lubin-Tate formal $\mathcal{O}_F$-modules. In §2, we define the various rings of power series that we use, and establish some of their properties. In

§3, we embed those rings in the usual rings of p-adic periods. In §4, we briefly survey Kisin and Ren's construction and explain why (φ_q, Γ_F)-modules over rings of power series in several variables are needed. In §5, we attach such objects to filtered φ_q-modules and prove theorem A. In §6, we prove theorem B. In §7, we study crystalline (φ_q, Γ_F)-modules and prove theorem C.

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1. Periods of Lubin-Tate formal groups

Let LT_h be the Lubin-Tate formal $\mathcal{O}_F$-module for which multiplication by p is given by $[p](T) = pT + T^q$. We denote by $[a](T)$ the element of $\mathcal{O}_F[\![T]\!]$ that gives the action of $a \in \mathcal{O}_F$ on LT_h and by $S(T,U) = T \oplus U$ the element of $\mathcal{O}_F[\![T,U]\!]$ that gives addition.

Let $\pi_0 = 0$ and for each $n \geqslant 1$, let $\pi_n \in \overline{\mathbf{Q}}_p$ be such that $[p](\pi_n) = \pi_{n-1}$, with $\pi_1 \neq 0$. We have $\mathrm{val}_p(\pi_n) = 1/q^{n-1}(q-1)$ if $n \geqslant 1$. Let $F_n = F(\pi_n)$ and let $F_\infty = \cup_{n \geqslant 1} F_n$. Recall that $\mathrm{Gal}(F_\infty/F) \simeq \mathcal{O}_F^\times$ and that the maximal abelian extension of F is $F_\infty \cdot F^{\mathrm{unr}}$. Denote by H_F the group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$, by Γ_F the group $\mathrm{Gal}(F_\infty/F)$ and by χ_{LT} the isomorphism $\chi_{\mathrm{LT}} : \Gamma_F \to \mathcal{O}_F^\times$. In the sequel, we sometimes directly identify Γ_F with $\mathcal{O}_F^\times$, that is we drop "χ_{LT}" from the notation to make the formulas less cumbersome.

Let $\widetilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^q} \mathcal{O}_{\mathbf{C}_p}/p$ and $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$ denote Fontaine's rings of periods (see [**Fon94**]). Note that we take the limit with respect to the maps $x \mapsto x^q$, which does not change the rings. Let $\varphi_q : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$ be given by $\varphi_q = \varphi^h$. Recall that in §9.2 of [**Col02**], Colmez has constructed a map $\{\cdot\} : \widetilde{\mathbf{E}}^+ \to \widetilde{\mathbf{A}}^+$ having the following property: if $x = (x_0, x_1, \ldots) \in \widetilde{\mathbf{E}}^+$, then $\{x\}$ is the unique element of $\widetilde{\mathbf{A}}^+$ that lifts x and satisfies $\varphi_q(\{x\}) = [p](\{x\})$. We then have $\theta(\{x\}) = \lim_{n \to \infty} [p^n](\widehat{x}_n)$, where $\widehat{x}_n \in \mathcal{O}_{\mathbf{C}_p}$ is any lift of x_n.

Let $u = \{(\pi_0, \pi_1, \ldots)\} \in \widetilde{\mathbf{A}}^+$, so that $g(u) = [\chi_{\mathrm{LT}}(g)](u)$ if $g \in G_F$. Let $\mathbf{A}_F$ be the p-adic completion of $\mathcal{O}_F[\![u]\!][1/u]$ inside $\widetilde{\mathbf{A}}^+$ and let $\mathbf{A}$ be the p-adic completion of $\mathbf{A}_F^{\mathrm{unr}}$. By the theory of the field of norms (see [**FW79**] and [**Win83**]), we have $\mathbf{A}_F = \mathbf{A}^{H_F}$.

Let $\log_{\mathrm{LT}}(T) \in F[\![T]\!]$ denote the Lubin-Tate logarithm map, which converges on the open unit disk and satisfies $\log_{\mathrm{LT}}([a](T)) = a \cdot \log_{\mathrm{LT}}(T)$ if $a \in \mathcal{O}_F$. Recall (see §9.3 of [**Col02**]) that $\log_{\mathrm{LT}}(u)$ converges in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ to an element t_F which satisfies $a(t_F) = a \cdot t_F$.

Let $Q_k(T)$ be the minimal polynomial of π_k over F. We have $Q_0(T) = T$, $Q_1(T) = p + T^{p-1}$ and $Q_{k+1}(T) = Q_k([p](T))$ if $k \geqslant 1$. Note that

$$\log_{\mathrm{LT}}(T) = T \cdot \prod_{k \geqslant 1} \frac{Q_k(T)}{p}.$$

Indeed, $\log_{\mathrm{LT}}(T) = \lim_{k\to\infty} p^{-k} \cdot [p^k](T)$ (§9.3 of [**Col02**]) and $[p^k](T) = Q_0(T) \cdots Q_k(T)$. Let $\exp_{\mathrm{LT}}(T)$ denote the inverse of $\log_{\mathrm{LT}}(T)$. We have $\exp_{\mathrm{LT}}(T) = \sum_{k=1}^{\infty} e_k T^k$ with $v_p(e_k) \geqslant -k/(q-1)$. For example, $\log_{\mathbf{G}_{\mathrm{m}}}(T) = \log(1+T)$ and $\exp_{\mathbf{G}_{\mathrm{m}}}(T) = \exp(T) - 1$.

Remark 1.1. — Our special choice of $[p](T) = pT + T^q$ is the simplest. Since $[p](T)$ belongs to $\mathbf{Z}_p[T]$, the series $Q_k(T)$, $\log_{\mathrm{LT}}(T)$ and $\exp_{\mathrm{LT}}(T)$ all have coefficients in $\mathbf{Q}_p$. It also implies that $[\sigma(a)](T) = \sigma([a](T))$, since $[a](T) = \exp_{\mathrm{LT}}(a \cdot \log_{\mathrm{LT}}(T))$.

Lemma 1.2. — *If $z \in \mathfrak{m}_{\mathbf{C}_p}$, then*

$$\frac{[1+a](z) - z}{a} = \log_{\mathrm{LT}}(z) \cdot \frac{dS}{dU}(z,0) + \mathrm{O}(a),$$

as $a \to 0$ in $\mathcal{O}_F$.

Proof. — We are looking at the limit of $(S(z,[a](z)) - z)/a$ as $a \to 0$. If a is small enough, then $[a](z) = \exp_{\mathrm{LT}}(a \cdot \log_{\mathrm{LT}}(z)) = a \cdot \log_{\mathrm{LT}}(z) + \mathrm{O}(a^2)$, which implies the lemma. $\square$

2. Rings of multivariable power series

We consider power series in the h variables $Y_0, \ldots, Y_{h-1}$. If $Y^m = Y_0^{m_0} \cdots Y_{h-1}^{m_{h-1}}$ is a monomial, then its weight is $w(m) = m_0 + pm_1 + \cdots + p^{h-1} m_{h-1}$. If I is a subinterval of $[0;+\infty]$ and if $J = \{j_1, \ldots, j_k\}$ is a subset of $\{0, \ldots, h-1\}$,, then (adapting Appendix A of [**Záb12**] to our situation) we define $\mathcal{R}^I(\{Y_j\}_{j \in J})$ to be the ring of power series

$$f(Y_{j_1}, \ldots, Y_{j_k}) = \sum_{m_1, \ldots, m_k \in \mathbf{Z}} a_{m_1 \ldots m_k} Y_{j_1}^{m_1} \cdots Y_{j_k}^{m_k},$$

such that $\mathrm{val}_p(a_m) + w(m)/r \to +\infty$ for all $r \in I$. In other words, $f(Y)$ is required to converge on the polyannulus $\{(Y_0, \ldots, Y_{h-1})$ such that $|Y_0| = p^{-1/r}, \ldots, |Y_{h-1}| = p^{-p^{h-1}/r}\}$ for all $r \in I$. We then define $W(f(Y), r) = \inf_{m \in \mathbf{Z}} (\mathrm{val}_p(a_m) + w(m)/r)$ and, if I is closed, $W(f(Y), I) = \inf_{r \in I} W(f(Y), r)$.

We let $\mathcal{R}^+(\{Y_j\}_{j \in J}) = \mathcal{R}^{[0;+\infty[}(\{Y_j\}_{j \in J})$ be the ring of holomorphic functions on the open unit polydisk corresponding to J. The Robba ring $\mathcal{R}(\{Y_j\}_{j \in J})$ is defined as $\mathcal{R}(\{Y_j\}_{j \in J}) = \cup_{r \geqslant 0} \mathcal{R}^{[r;+\infty[}(\{Y_j\}_{j \in J})$. In order to lighten the notation, we write $\mathcal{R}^I(Y)$, $\mathcal{R}^+(Y)$ and $\mathcal{R}(Y)$ instead of $\mathcal{R}^I(Y_0, \ldots, Y_{h-1})$, $\mathcal{R}^+(Y_0, \ldots, Y_{h-1})$ and $\mathcal{R}(Y_0, \ldots, Y_{h-1})$.

The rings $\mathcal{R}^I(\{Y_j\}_{j\in J})$ are endowed with an F-linear action of Γ_F, given by the formula $a(Y_j)=[\sigma^j(a)](Y_j)$. There is also an F-linear Frobenius map :

$$\varphi_q : \mathcal{R}^I(\{Y_j\}_{j\in J}) \to \mathcal{R}^{I'}(\{Y_j\}_{j\in J}),$$

given by $Y_j \mapsto [p](Y_j)$, where I' is the image of I by the map $r \mapsto \min(r/q, r/(r+1))$.

On the ring $\mathcal{R}^I(Y)$, we can define in addition an absolute σ-semilinear Frobenius map φ by $Y_j \mapsto Y_{j+1}$ for $0 \leqslant j \leqslant h-2$ and $Y_{h-1} \mapsto [p](Y_0)$. This map φ has the property that $\varphi^h=\varphi_q$, and it also commutes with Γ_F.

Let $t_i=\log_{\mathrm{LT}}(Y_i)$. Since $a(Y_i)=[\sigma^i(a)](Y_i)$ if $a\in\Gamma_F$, we have $a(t_i)=\sigma^i(a)\cdot t_i$. If $t=t_0\cdots t_{h-1}$, then $g(t)=\mathrm{N}_{F/\mathbf{Q}_p}(\chi_{\mathrm{LT}}(g))\cdot t=\chi_{\mathrm{cycl}}(g)\cdot t$ if $g\in G_F$ and $\varphi(t)=pt$ so that t behaves like a multiple of the "usual" t of p-adic Hodge theory (see proposition 3.4 for a more precise statement).

The following two propositions are variations on the "Weierstrass division theorem".

Proposition 2.1. — *Let $I=[0;s]$ or $[0;s[$ and let $P(T)\in\mathcal{O}_F[T]$ be a monic polynomial of degree d whose nonleading coefficients are all divisible by p. If $f\in\mathcal{R}^I(\{Y_j\}_{j\in J})$, then there exists $g\in\mathcal{R}^I(\{Y_j\}_{j\in J})$ and $f_0,\ldots,f_{d-1}\in\mathcal{R}^I(\{Y_j\}_{j\in J\setminus\{i\}})$ such that $f=f_0+f_1Y_i+\cdots+f_{d-1}Y_i^{d-1}+gP(Y_i)$.*

Proof. — If $I=[0;s]$ is closed, then this is a straightforward consequence of the Weierstrass division theorem. Since g and the f_i's are uniquely determined, the result extends to the case when $I=[0;s[$. $\square$

Proposition 2.2. — *Let $I=[s;s]$ and let $P(T)\in\mathcal{O}_F[T]$ be a monic polynomial of degree d, all of whose roots are of valuation $-1/s$. If $f\in\mathcal{R}^I(\{Y_j\}_{j\in J})$, then there exists $g\in\mathcal{R}^I(\{Y_j\}_{j\in J})$ and $f_0,\ldots,f_{d-1}\in\mathcal{R}^I(\{Y_j\}_{j\in J\setminus\{i\}})$ such that $f=f_0+f_1Y_i+\cdots+f_{d-1}Y_i^{d-1}+gP(Y_i)$.*

Proof. — The polynomial $Q(T)=P(1/T)T^d/P(0)$ is monic and all its roots are of valuation $1/s$. Write $f=f^++f^-$ where f^+ contains positive powers of Y_i and f^- contains negative powers of Y_i. One may Weierstrass divide f^+ by $P(Y_i)$ and f^- by $Q(1/Y_i)$, which implies the proposition. $\square$

Lemma 2.3. — *The action of Γ_F on $\mathcal{R}(Y)$ is $\mathbf{Q}_p$-analytic, and we have*

$$[1+a](f(Y)) = f(Y) + \sum_{j=0}^{h-1} \sigma^j(a)\cdot\log_{\mathrm{LT}}(Y_j)\cdot\frac{dS}{dU}(Y_j,0)\cdot\frac{df}{dY_j}(Y)+\mathrm{O}(a^2).$$

Proof. — This follows from $[1+a](Y_j)=Y_j\oplus[a](Y_j)=Y_j\oplus(\sigma^j(a)\cdot\log_{\mathrm{LT}}(Y_j)+\mathrm{O}(a^2))$. $\square$

Proposition 2.4. — *Let $\rho = (\rho_1, \dots, \rho_{h-1})$ and let $\mathcal{R}^{\rho}_{F_k}(T_1, \dots, T_{h-1})$ denote the ring of Laurent series converging for $|T_i| = \rho_i$, with coefficients in F_k. If the $z_i \in \mathfrak{m}_{\widehat{F}_\infty}$ are such that $\log_{\mathrm{LT}}(z_i) \neq 0$, $|z_i| = \rho_i$ and $g(z_i) = [\sigma^i(g)](z_i)$ for $g \in \mathcal{O}_F^\times$, then the map $\mathcal{R}^{\rho}_{F_k}(T_1, \dots, T_{h-1}) \to \mathbf{C}_p$ given by evaluating at $(z_1, \dots, z_{h-1})$ is injective.*

Proof. — Suppose that $f(z_1, \dots, z_{h-1}) = 0$ for some $f \in \mathcal{R}^{\rho}_{F_k}(T_1, \dots, T_{h-1})$. If $g \in \Gamma_{F_k}$, then $f(g(z_1), \dots, g(z_{h-1})) = 0$. If $g = 1 + a$ with a small, then lemma 1.2 provides us with $h-1$ elements $y_1, \dots, y_{h-1}$ of $\widehat{F}_\infty$ such that $g(z_i) = z_i + \sigma^i(a) \cdot y_i + \mathrm{O}(a^2)$. Since $y_i = \log_{\mathrm{LT}}(z_i) \cdot dS/dU(z_i, 0)$ and dS/dU is a unit and $\log_{\mathrm{LT}}(z_i) \neq 0$, the elements $y_1, \dots, y_{h-1}$ are all nonzero.

If $f \neq 0$ and m is the smallest index for which f has a nonzero partial derivative of order m at $(z_1, \dots, z_{h-1})$ and if we expand $f(g(z_1), \dots, g(z_{h-1}))$ around $(z_1, \dots, z_{h-1})$ (which generalizes lemma 2.3), we get

$$\sum_{j_1 + \cdots + j_{h-1} = m} (\sigma^1(a) y_1)^{j_1} \cdots (\sigma^{h-1}(a) y_{h-1})^{j_1} \frac{d^m f}{dT_1^{j_1} \cdots dT_{h-1}^{j_{h-1}}}(z_1, \dots, z_{h-1}) + \mathrm{O}(a^{m+1}).$$

Since $f(g(z_1), \dots, g(z_{h-1})) = 0$, the above linear combination is a homogeneous polynomial, of degree m in $h-1$ variables and coefficients in $\widehat{F}_\infty$, that is identically zero on $(\sigma^1(a), \dots, \sigma^{h-1}(a))$. The shortest nonzero polynomial that is identically zero on $(\sigma^1(a), \dots, \sigma^{h-1}(a))$ can be taken to have coefficients in F and Artin's theorem on the algebraic independence of characters implies that it is equal to zero. Since all the y_i's are nonzero, all the partial derivatives of order m of f are zero, so that finally $f = 0$. $\square$

3. Embeddings in $\mathbf{B}_{\mathrm{dR}}$

We now explain how to embed the rings of power series of the previous section in the usual rings of p-adic periods. Let $\widetilde{\mathbf{B}}^I$ be the ring defined in §2.1 of [**Ber02**]. This ring is complete with respect to the valuation $V(\cdot, I)$ (denoted by $V_I(\cdot)$ in §2.1 of ibid.). Recall that if $x = \sum_{k \geqslant 0} p^k [x_k] \in \widetilde{\mathbf{A}}^+$, then $V(x, r) = \inf_k(\mathrm{val}_{\mathbf{E}}(x_k) + krp/(p-1))$. Set $r_F = p^{h-1} \cdot q/(q-1) \cdot (p-1)/p$ (for example, $r_{\mathbf{Q}_p} = 1$ and if $h > 1$, then $r_F < p^{h-1}$).

Proposition 3.1. — *If $r \geqslant r_F$, then $V(\varphi^j(u), r) = p^j \cdot q/(q-1)$ for $0 \leqslant j \leqslant h-1$.*

Proof. — Recall that $u = \{\pi\}$ where $\pi = (\pi_0, \pi_1, \dots)$ with $\mathrm{val}_p(\pi_n) = 1/q^{n-1}(q-1)$ for $n \geqslant 1$, so that $\mathrm{val}_{\mathbf{E}}(\pi) = q/(q-1)$. We have $\varphi^j(u) = [\pi^{p^j}] + \sum_{k \geqslant 1} p^k [u_{k,j}]$ where $\mathrm{val}_{\mathbf{E}}(u_{k,j}) > 0$, so that if $r \geqslant r_F$, then $\varphi^j(u)/[\pi^{p^j}]$ is a unit of $\widetilde{\mathbf{A}}^{\dagger, r}$ and the proposition follows. $\square$

Note that a better estimate on the $\mathrm{val}_{\mathbf{E}}(u_{k,j})$ would give a better estimate of r_F. Let $s_n = p^{n-h}(q-1)$ and let $r_n = p^{n-1}(p-1)$ (so that $s_n \cdot q/(q-1) = r_n \cdot p/(p-1)$).

Proposition 3.2. — *If $n \geqslant h$, and if $f(Y) \in \mathcal{R}^{[s_n;s_n]}(Y)$, then $f(u, \dots, \varphi^{h-1}(u))$ converges in $\widetilde{\mathbf{B}}^{[r_n;r_n]}$.*

Proof. — If $f(Y) = \sum_{m \in \mathbf{Z}^h} a_m Y^m \in \mathcal{R}^{[s_n;s_n]}(Y)$, then $\mathrm{val}_p(a_m) + w(m)/(p^{n-h}(q-1)) \to +\infty$. If $n \geqslant h$, then $r_n > r_F$ so that $V(\varphi^j(u), r) = p^j \cdot q/(q-1)$ for $0 \leqslant j \leqslant h-1$ by proposition 3.1, and then

$$V(a_{m_0,\dots,m_{h-1}} u^{m_0} \cdots \varphi^{h-1}(u)^{m_{h-1}}, r_n) \to +\infty.$$

The series $f(u, \dots, \varphi^{h-1}(u))$ therefore converges in $\widetilde{\mathbf{B}}^{[r_n;r_n]}$. ☐

Corollary 3.3. — *If $n \geqslant h$, and if $f(Y) \in \mathcal{R}^{[0;s_n]}(Y)$, then $f(u, \dots, \varphi^{h-1}(u))$ converges in $\widetilde{\mathbf{B}}^{[0;r_n]}$. If $f(Y) \in \mathcal{R}^{+}(Y)$, then $f(u, \dots, \varphi^{h-1}(u))$ converges in $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$.*

Proof. — If $f \in \mathcal{R}^{[0;s_n]}(Y)$, then each term of the series $f(u, \dots, \varphi^{h-1}(u))$ belongs to $\widetilde{\mathbf{B}}^{+}$ so that it converges in $\widetilde{\mathbf{B}}^{[0;r_n]}$ by the maximum modulus principle (corollary 2.20 of [**Ber02**]). The second assertion follows by passing to the limit. ☐

Proposition 3.4. — *The image of $\log_{\mathrm{LT}}(Y_0) \cdots \log_{\mathrm{LT}}(Y_{h-1})$ under the above map belongs to $p^{h-1} \cdot \mathbf{Z}_p^{\times} \cdot t$, where t is the usual t of p-adic Hodge theory.*

Proof. — The product $\log_{\mathrm{LT}}(Y_0) \cdots \log_{\mathrm{LT}}(Y_{h-1})$ is an element v of $\mathbf{B}^{+}_{\mathrm{dR}}$ that satisfies $g(v) = \chi_{\mathrm{cycl}}(g) \cdot v$ so that it is a multiple of t, and we need to determine the p-adic valuation of the multiplying coefficient. By Boxall's theorem (theorem B of [**Box86**]), we have $\mathrm{val}_p(\theta(t/\log_{\mathrm{LT}}(u))) = 1/(p-1) - 1/(q-1)$. On the other hand, $\mathrm{val}_p(\log_{\mathrm{LT}}(\varphi^j(u))) = \mathrm{val}_p(\lim_{n\to\infty}[p^n](\pi_n^{p^j})) = 1 + p^j/(q-1)$ if $1 \leqslant j \leqslant h-1$. This implies that $\mathrm{val}_p(t/\log_{\mathrm{LT}}(u) \cdots \log_{\mathrm{LT}}(\varphi^{h-1}(u))) = -(h-1)$, and hence the proposition. ☐

Definition 3.5. — Let $\iota_n : \mathcal{R}^{[s_n;s_n]}(Y) \to \mathbf{B}^{+}_{\mathrm{dR}}$ be the compositum of the map defined above, with the map $\varphi^{-n} : \widetilde{\mathbf{B}}^{[r_n;r_n]} \to \widetilde{\mathbf{B}}^{[r_0;r_0]}$ and the map $\widetilde{\mathbf{B}}^{[r_0;r_0]} \subset \mathbf{B}^{+}_{\mathrm{dR}}$ defined in §2.2 of [**Ber02**].

It follows from the definition as well as the formulas for φ and the action of Γ_F on $\mathcal{R}^I(Y)$ that $\iota_{n+1}(\varphi(f)) = \iota_n(f)$ when applicable, and that $g(\iota_n(f)) = \iota_n(g(f))$ if $g \in G_F$. Since $\iota_n(t) = p^{-n}t$, we can extend ι_n to $\iota_n : \mathcal{R}^{[s_n;s_n]}(Y)[1/t] \to \mathbf{B}_{\mathrm{dR}}$.

Theorem 3.6. — *If $n \geqslant h$, if $f \in \mathcal{R}^{[s_n;s_n]}(Y)$, and if $n = hk + i$ with $0 \leqslant i \leqslant h-1$, then we have $\iota_n(f) \in \mathrm{Fil}^1 \mathbf{B}^{+}_{\mathrm{dR}}$ if and only if $f \in Q_k(Y_i) \cdot \mathcal{R}^{[s_n;s_n]}(Y)$.*

Proof. — Recall that $u = \{(\pi_0, \pi_1, \dots)\} \in \widetilde{\mathbf{A}}^{+}$. If $m \geqslant 1$ and we define $u_m = \theta(\varphi^{-m}(u)) \in \widehat{F}_\infty$, then $g(u_m) = [\sigma^{-m}(g)](u_m)$. Note that if $m = h\ell$, then $u_m = \theta(\varphi_q^{-\ell}(u)) = \pi_\ell$. The theorem is equivalent to the assertion that $f^{\sigma^{-n}}(u_n, \dots, u_{n-h+1}) = 0$ in $\mathbf{C}_p$ if and only if

$f \in Q_k(Y_i) \cdot \mathcal{R}^{[s_n;s_n]}(Y)$. We have $u_{n-i} = \pi_k$ so that if f belongs to $Q_k(Y_i) \cdot \mathcal{R}^{[s_n;s_n]}(Y)$, then $f^{\sigma^{-n}}(u_n, \dots, u_{n-h+1}) = 0$.

Since $Q_k(T)$ is a monic polynomial of degree $d = q^{k-1}(q-1)$, whose nonleading coefficient is divisible by p, we can use proposition 2.2 to write $f^{\sigma^{-n}} = f_0 + Y_i f_1 + \cdots + Y_i^{d-1} f_{d-1} + Q_k(Y_i) r$ with f_i a power series in the Y_j's with $j \neq i$. Proposition 2.4 applied to $g = f_0 + \pi_k f_1 + \cdots + \pi_k^{d-1} f_{d-1}$, with the T_j's a suitable permutation of the Y_j's, shows that $g = 0$. Therefore, $f = Q_k(Y_i) r^{\sigma^n}$, which proves the theorem. □

Corollary 3.7. — *If $n \geqslant h$, then the map $\iota_n : \mathcal{R}^{[s_n;s_n]}(Y) \to \mathbf{B}_{\mathrm{dR}}^+$ is injective. If $n \in \mathbf{Z}$, then the map $\iota_n : \mathcal{R}^+(Y) \to \mathbf{B}_{\mathrm{dR}}^+$ is injective.*

Proof. — The first assertion follows from theorem 3.6. The second follows from that, and from the fact that $\iota_{n+1}(\varphi(f)) = \iota_n(f)$ for the other n. □

4. (φ_q, Γ_F)-modules in one variable

Before constructing (φ_q, Γ_F)-modules over $\mathcal{R}(Y)$, we review Kisin and Ren's construction of (φ_q, Γ_F)-modules in one variable and explain why we need rings in several variables.

Let Y_0 be the variable of §2, and let $\mathcal{E}(Y_0)$ be Fontaine's field of [**Fon90**] with coefficients in F, that is $\mathcal{E}(Y_0) = \mathcal{O}_{\mathcal{E}}(Y_0)[1/p]$ where $\mathcal{O}_{\mathcal{E}}(Y_0)$ is the p-adic completion of $\mathcal{O}_F[\![Y_0]\!][1/Y_0]$. We let $\mathcal{E}^\dagger(Y_0)$ and $\mathcal{R}(Y_0)$ denote the corresponding overconvergent and Robba rings. If I is a subinterval of $[0;+\infty]$, then we denote as above by $\mathcal{R}^I(Y_0)$ the set of power series $f(Y) = \sum_{m \in \mathbf{Z}} a_m Y_0^m$ that belong to $\mathcal{R}^I(Y_0, \dots, Y_{h-1})$ via the natural inclusion.

If K/F is a finite extension, then by the theory of the field of norms, there corresponds to it a finite extension $\mathcal{E}_K(Y_0)$ of $\mathcal{E}(Y_0)$, of degree $[K_\infty : F_\infty]$. A (φ_q, Γ_K)-module over $\mathcal{E}_K(Y_0)$ is a finite dimensional $\mathcal{E}_K(Y_0)$-vector space D, along with a semilinear φ_q and a compatible action of Γ_K. We say that D is étale if $\mathrm{D} = \mathcal{E}_K(Y_0) \otimes_{\mathcal{O}_{\mathcal{E}_K}(Y_0)} \mathrm{D}_0$ where D_0 is a (φ_q, Γ_K)-module over $\mathcal{O}_{\mathcal{E}_K}(Y_0)$. By specializing the constructions of [**Fon90**], Kisin and Ren prove the following theorem in their paper (theorem 1.6 of [**KR09**]).

Theorem 4.1. — *The functors*

$$V \mapsto (\widehat{\mathcal{E}}(Y_0)^{\mathrm{unr}} \otimes_F V)^{H_K} \text{ and } \mathrm{D} \mapsto (\widehat{\mathcal{E}}(Y_0)^{\mathrm{unr}} \otimes_{\mathcal{E}_K(Y_0)} \mathrm{D})^{\varphi_q = 1}$$

give rise to mutually inverse equivalences of categories between the category of F-linear representations of G_K and the category of étale (φ_q, Γ_K)-modules over $\mathcal{E}_K(Y_0)$.

We say that an F-linear representation of G_K is F-analytic if it is Hodge-Tate with weights 0 at all embeddings $\tau \neq \mathrm{Id}$. Kisin and Ren then go on to show that if $K \subset F_\infty$, and if V is a crystalline representation of G_K, that is F-analytic, then the (φ_q, Γ_K)-module attached to V is overconvergent (see §3.3 of ibid.).

Assume from now on that $K \subset F_\infty$, so that $\mathcal{E}_K(Y_0) = \mathcal{E}(Y_0)$. If D is a (φ_q, Γ_K)-module over $\mathcal{R}(Y_0)$, and if $g \in \Gamma_K$ is close enough to 1, then by standard arguments (see §4.1 of [**Ber02**] or §2.1 of [**KR09**]), the series $\log(g) = \log(1+(g-1))$ gives rise to a differential operator $\nabla_g : \mathrm{D} \to \mathrm{D}$. The map $\operatorname{Lie}\Gamma_F \to \operatorname{End}(\mathrm{D})$ arising from $v \mapsto \nabla_{\exp(v)}$ is $\mathbf{Q}_p$-linear, and we say that D is F-analytic if this map is F-linear (see §2.1 of [**KR09**] and §1.3 of [**FX12**]).

Theorem 4.2. — *If V is an overconvergent F-linear representation of G_K, and if $\mathrm{D}(V) = \mathcal{R}(Y_0) \otimes_{\mathcal{E}^\dagger(Y_0)} \mathrm{D}^\dagger(V)$ is F-analytic, then V is F-analytic.*

Proof. — Choose $1 \leqslant j \leqslant h-1$, and let $n \gg 0$ be such that $n = -j \bmod h$. By proposition 3.2, we have a map $\theta \circ \varphi^{-n} : \mathcal{R}^{[s_n;s_n]}(Y_0) \to \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$, which gives rise to an isomorphism

$$\mathbf{C}_p \otimes^{\theta\circ\varphi^{-n}}_{\mathcal{R}^{[s_n;s_n]}(Y_0)} \mathrm{D}^{[s_n;s_n]}(V) \to \mathbf{C}_p \otimes_F^{\sigma^j} V.$$

Let $\widehat{F}_\infty^{(j)}$ denote the subfield of locally σ^j-analytic vectors of $\widehat{F}_\infty$ for the action of Γ_K. Note that $\theta \circ \varphi^{-n}(\mathcal{R}^{[s_n;s_n]}(Y_0)) \subset \widehat{F}_\infty^{(j)}$. Let $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$ be the $\widehat{F}_\infty^{(j)}$-vector space

$$\mathrm{D}_{\mathrm{Sen}}^{(j)}(V) = \widehat{F}_\infty^{(j)} \otimes_{\theta\circ\varphi^{-n}(\mathcal{R}^{[s_n;s_n]}(Y_0))} \theta \circ \varphi^{-n}(\mathrm{D}^{[s_n;s_n]}(V)).$$

It is of dimension d, its image in $(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F}$ generates $\mathbf{C}_p \otimes_F^{\sigma^j} V$, and its elements are all locally σ^j-analytic vectors of $(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F}$ because $\mathrm{D}(V)$ is F-analytic. If $y \in \mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, then $(g(y)-y)/(\sigma^j(g)-1)$ has a limit as $g \to 1$, and we call $\nabla(y)$ this limit. We then have $g(y) = \exp(\log_p(\sigma^j \circ \chi_{\mathrm{LT}}(g)) \cdot \nabla)(y)$ if $g \in \Gamma_K$ is close to 1.

Recall that there exists $a_j \in \mathbf{C}_p$ such that $\log_p(\sigma^j \circ \chi_{\mathrm{LT}}(g)) = g(a_j) - a_j$. For example, one can take $a_j = \log_p \theta(t_j)$. The element a_j then belongs to $\widehat{F}_\infty^{(j)}$ for obvious reasons. If $y \in \mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, then $\exp(-p^m a_j \nabla)(y)$ converges for $m = m(y)$ large enough, and its limit is an element of $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$ that is fixed by Γ_{K_m}.

If $y_1, \ldots, y_d$ is a basis of $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, and if $m \geqslant \max m(y_i)$, then the $\exp(-p^m a_j \nabla)(y_i)$ are fixed by Γ_{K_m} and linearly independent over K_m. This implies that the restriction of V to G_{K_m} is $\mathbf{C}_p$-admissible for the embedding σ^j. This is true for all $1 \leqslant j \leqslant h-1$, and therefore V is F-analytic. □

Note that an analogous argument for the proof of theorem 4.2 was recently given by Bingyong Xie.

Corollary 4.3. — *If V is an absolutely irreducible F-linear overconvergent representation of G_K, then there exists a character δ of Γ_K such that $V \otimes \delta$ is F-analytic.*

Proof. — We give a sketch of the proof. Choose some $g \in \Gamma_K$ such that $\log_p(\chi_{\mathrm{LT}}(g)) \neq 0$, and let $\nabla = \log(g)/\log_p(\chi_{\mathrm{LT}}(g))$. Choose $r > 0$ large enough and $s \geqslant qr$. If $a \in \mathcal{O}_F$, and if $\mathrm{val}_p(a) \geqslant n$ for $n = n(r,s)$ large enough, then the series $\exp(a \cdot \nabla)$ converges to an

operator on $\mathrm{D}^{[r;s]}(V)$. This way, we can define a twisted action of Γ_{K_n} on $\mathrm{D}^{[r;s]}(V)$, by the formula $h \star x = \exp(\log_p(\chi_{\mathrm{LT}}(h)) \cdot \nabla)(x)$. This action is now F-analytic by construction.

Since $s \geqslant qr$, the modules $\mathrm{D}^{[q^m r;q^m s]}(V)$ for $m \geqslant 0$ are glued together by φ_q and this way, we get a new action of Γ_{K_n} on $\mathrm{D}(V)$. Since φ_q is unchanged, this new $(\varphi_q, \Gamma_{K_n})$-module is étale, and therefore corresponds to a representation W of G_{K_n}. This representation W is F-analytic by theorem 4.2, and its restriction to H_F is isomorphic to V.

The space $\mathrm{Hom}(V, \mathrm{ind}_{G_{K_n}}^{G_K} W)^{H_F}$ is nonempty, and is a finite dimensional representation of Γ_K. Since Γ_K is abelian, we find (possibly extending scalars) a character δ of Γ_K and a nonzero $f \in \mathrm{Hom}(V, \mathrm{ind}_{G_{K_n}}^{G_K} W)^{H_F}$ such that $h(f) = \delta(h) \cdot f$ if $h \in G_K$. This f gives rise to a nonzero G_K-equivariant map $V \otimes \delta \to \mathrm{ind}_{G_{K_n}}^{G_K} W$. Since $\mathrm{ind}_{G_{K_n}}^{G_K} W$ is F-analytic and V is absolutely irreducible, the corollary follows. □

Corollary 4.3 suggests that if we want to attach overconvergent (φ_q, Γ_K)-modules to all F-linear representations of G_K, then we need to go beyond the objects in only one variable. We finish with a conjecture that seems reasonable enough, since it holds for crystalline representations by the work of Kisin and Ren. See also theorem 0.3 of [**FX12**].

Conjecture 4.4. — *If V is F-analytic, then it is overconvergent.*

5. Construction of $\mathcal{R}^+(Y)$-modules

We now explain how to construct some free $\mathcal{R}^+(Y)$-modules $\mathrm{M}^+(D)$ of finite rank that are attached to filtered φ_q-modules D. Let D be a finite dimensional F-vector space, endowed with an F-linear Frobenius map $\varphi_q : D \to D$, and an action of G_F on D that factors through Γ_F and commutes with φ_q.

If $n \in \mathbf{Z}$, let $\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^n} D$ denote the tensor product of $\mathbf{B}_{\mathrm{dR}}$ and D above F, where F maps to $\mathbf{B}_{\mathrm{dR}}$ via φ^n. We then have $b \otimes a \cdot d = \varphi^n(a) \cdot b \otimes d$. Note that $\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^n} D$ only depends on $n \bmod h$. For each $0 \leqslant j \leqslant h-1$, let $\mathrm{Fil}_j^\bullet$ be a filtration on D that is stable under Γ_F, and define $W_{\mathrm{dR}}^{+,j}(D) = \mathrm{Fil}_j^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^j} D)$ so that $W_{\mathrm{dR}}^{+,j}$ is a G_F-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattice of $\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^j} D$.

Example 5.1. — If V is an F-linear crystalline representation of G_F of dimension d, then $\mathrm{D}_{\mathrm{cris}}(V)$ is a free $F \otimes_{\mathbf{Q}_p} F$-module of rank d and we have $\mathrm{D}_{\mathrm{cris}}(V) = D \oplus \varphi(D) \oplus \cdots \oplus \varphi^{h-1}(D)$, according to the decomposition of $F \otimes_{\mathbf{Q}_p} F$ as $\prod_{\sigma^i : F \to F} F$. Each $\varphi^j(D)$ has the filtration induced from $\mathrm{D}_{\mathrm{cris}}(V)$, and we set $\mathrm{Fil}_j^k D = \varphi^{-j}(\mathrm{Fil}^k \mathrm{D}_{\mathrm{cris}}(V) \cap \varphi^j(D))$.

Let $\lambda_j = \log_{\mathrm{LT}}(Y_j)/Y_j$ and $\lambda = \lambda_0 \cdots \lambda_{h-1}$, so that t is a $\mathbf{Q}_p$-multiple of $\lambda \cdot Y_0 \cdots Y_{h-1}$. If $y = \sum_i f_i \otimes d_i \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D$, let $\iota_n(y) = \sum_i \iota_n(f_i) \otimes d_i \in \mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^{-n}} D$.

Definition 5.2. — Let $\mathrm{M}^+(D)$ be the set of $y \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D$ that satisfy $\iota_k(y) \in W_{\mathrm{dR}}^{+,-k}(D)$ for all $k \geqslant h$.

In the remainder of this section, we prove that $\mathrm{M}^+(D)$ is a free $\mathcal{R}^+(Y)$-module of rank $\dim D$. Let $n = hm + (h-1)$ where $m \geqslant 1$. Recall that on the ring $\mathcal{R}^{[0;s_n]}(Y)$, the map ι_k is defined for $h \leqslant k \leqslant n$. Let

$$\mathrm{M}(D)^{[0;s_n]} = \{y \in \mathcal{R}^{[0;s_n]}(Y)[1/\lambda] \otimes_F D,\ \iota_k(y) \in W_{\mathrm{dR}}^{+,-k}(D) \text{ for all } h \leqslant k \leqslant n\}.$$

For $0 \leqslant j \leqslant h-1$, let

$$N_j^{[0;s_n]} = \{y \in \mathcal{R}^{[0;s_n]}(Y_j)[1/\lambda_j] \otimes_F D,\ \varphi_q^{-k}\varphi^{-j}(y) \in W_{\mathrm{dR}}^{+,-j}(D) \text{ for all } 1 \leqslant k \leqslant m\}.$$

Proposition 5.3. — *The $\mathcal{R}^{[0;s_n]}(Y_j)$-module $N_j^{[0;s_n]}$ is free of rank $d = \dim D$, and the map $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)}^{\varphi_q^{-k}\varphi^{-j}} N_j^{[0;s_n]} \to W_{\mathrm{dR}}^{+,-j}(D)$ is an isomorphism for all $1 \leqslant k \leqslant m$.*

Proof. — Since there is only one variable, the proof is a standard argument, analogous to the one which one can find in §II.1 of [**Ber08b**] or §2.2 of [**KR09**]. □

Let $M_j^{[0;s_n]} = \mathcal{R}^{[0;s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}$, where $f_j = \lambda/\lambda_j$.

Proposition 5.4. — *We have $\mathrm{M}(D)^{[0;s_n]}[1/f_j] = M_j^{[0;s_n]}$.*

Proof. — In the sequel, we use the fact that $Q_1(Y_j)\cdots Q_m(Y_j)$ and λ_j generate the same ideal of $\mathcal{R}^{[0;s_n]}(Y_j)$ (recall that $n = hm + (h-1)$). Let a and b be two integers such that

$$t^a \mathbf{B}_{\mathrm{dR}}^+ \otimes_F^{\varphi^j} D \subset W_{\mathrm{dR}}^{+,j}(D) \subset t^{-b}\mathbf{B}_{\mathrm{dR}}^+ \otimes_F^{\varphi^j} D,$$

for all j. We then have $\mathrm{M}(D)^{[0;s_n]} \subset \lambda^{-b} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D$ by theorem 3.6.

We have $\varphi^{-(hk+j)}(\mathcal{R}^{[0;s_n]}(Y)[1/f_j]) \subset \mathbf{B}_{\mathrm{dR}}^+$ for all $1 \leqslant k \leqslant m$ so that if $y \in M_j^{[0;s_n]}$, then $\varphi^{-(hk+j)}(y) \in W_{\mathrm{dR}}^{+,-j}(D)$ for all $1 \leqslant k \leqslant m$. On the other hand, if $y \in M_j^{[0;s_n]}$, then $y \in \lambda^{-c} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D$ for some $c \geqslant 0$, so that $f_j^{a+c} y \in \mathrm{M}(D)^{[0;s_n]}$. This implies that $M_j^{[0;s_n]} \subset \mathrm{M}(D)^{[0;s_n]}[1/f_j]$.

We now prove that $\mathrm{M}(D)^{[0;s_n]} \subset M_j^{[0;s_n]}$. Choose $y \in \mathrm{M}(D)^{[0;s_n]}$. Since

$$\mathrm{M}(D)^{[0;s_n]} \subset \lambda^{-b} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D,$$

we can write $y = \lambda^{-b} \sum_k f_k \otimes d_k$. By Weierstrass dividing (proposition 2.1) the f_k's by the polynomial $(Q_1(Y_j)\cdots Q_m(Y_j))^{a+b}$, we can write $y = (Q_1(Y_j)\cdots Q_m(Y_j))^{a+b} z + y_0$ with $y_0 \in \mathcal{R}^{[0;s_n]}(Y)[1/\lambda] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}$. Note that $(Q_1(Y_j)\cdots Q_m(Y_j))^{a+b} z \in M_j^{[0;s_n]}$ because $t^a \mathbf{B}_{\mathrm{dR}}^+ \otimes_F^{\varphi^j} D \subset W_{\mathrm{dR}}^{+,j}(D)$, so that $(Q_1(Y_j)\cdots Q_m(Y_j))^a \cdot D \subset N_j^{[0;s_n]}$.

Write $y_0 = \sum_{k=1}^d a_k \otimes n_k$ where $a_k \in \mathcal{R}^{[0;s_n]}(Y)[1/\lambda]$ and $n_1, \ldots, n_d$ is a basis of $N_j^{[0;s_n]}$. The element y_0 satisfies $\varphi_q^{-\ell}\varphi^{-j}(y_0) \in W_{\mathrm{dR}}^{+,-j}(D)$ for all $1 \leqslant \ell \leqslant m$. By proposition 5.3, the map

$$\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)}^{\varphi_q^{-\ell}\varphi^{-j}} N_j^{[0;s_n]} \to W_{\mathrm{dR}}^{+,-j}(D)$$

is an isomorphism; this implies that $\varphi_q^{-\ell}\varphi^{-j}(a_k) \in \mathbf{B}_{\mathrm{dR}}^+$ for all $1 \leqslant \ell \leqslant m$. Theorem 3.6 now implies that a_k has no pole at any of the $Q_1(Y_j), \dots, Q_m(Y_j)$, so that we have $a_k \in \mathcal{R}^{[0;s_n]}(Y)[1/f_j]$. This implies that $y_0 \in M_j^{[0;s_n]}$, and therefore also y.

This proves that $\mathrm{M}(D)^{[0;s_n]} \subset M_j^{[0;s_n]}$ and therefore $\mathrm{M}(D)^{[0;s_n]}[1/f_j] = M_j^{[0;s_n]}$. □

Theorem 5.5. — *The $\mathcal{R}^+(Y)$-module $\mathrm{M}^+(D)$ is free of rank $\dim D$.*

Proof. — Note that $f_0, \dots, f_{h-1}$ generate the unit ideal of $\mathcal{R}^{[0;s_n]}(Y)$. Since the $\mathcal{R}^{[0;s_n]}(Y)[1/f_j]$-module $\mathrm{M}(D)^{[0;s_n]}[1/f_j] = M_j^{[0;s_n]}$ is free of rank $\dim D$ for all j, we have (see theorem 1 of II.5.2 of [**Bou61**]) that $\mathrm{M}(D)^{[0;s_n]}$ is projective over $\mathcal{R}^{[0;s_n]}(Y)$ (note that by Kedlaya's "Quillen-Suslin theorem for Tate algebras", proposition 6.6 of [**Ked04**], $\mathrm{M}(D)^{[0;s_n]}$ is then actually free over $\mathcal{R}^{[0;s_n]}(Y)$).

The system $\{\mathrm{M}(D)^{[0;s_{hm+(h-1)}]}\}_{m\geqslant 1}$ forms a vector bundle over the open unit polydisk, whose global sections are precisely $\mathrm{M}^+(D)$. By combining proposition 2 on page 87 of [**Gru68**] (note that his "A_m" is defined at the bottom of page 82 of loc. cit.), and the main theorem of [**Bar81**], $\mathrm{M}^+(D)$ is free of rank $\dim D$ over $\mathcal{R}^+(Y)$. □

6. Properties of $\mathrm{M}^+(D)$

We now prove that $\mathcal{R}(Y) \otimes_{\mathcal{R}^+(D)} \mathrm{M}^+(D)$ is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, and that if D arises from a crystalline representation V, then $\mathrm{M}^+(D)$ and V are naturally related. It is clear from the definition that $\mathrm{M}^+(D)$ is stable under the action of Γ_F. We also have $\lambda^a \cdot \mathcal{R}^+(Y) \otimes_F D \subset \mathrm{M}^+(D)$ so that

$$\mathcal{R}^+(Y)[1/\lambda] \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) = \mathcal{R}^+(Y)[1/\lambda] \otimes_F D.$$

Recall that $n = hm + (h-1)$ with $m \geqslant 1$.

Lemma 6.1. — *The $\mathcal{R}^{[0;s_n]}(Y_j)$-module $N_j^{[0;s_n]}$ is stable under φ_q, and $N_j^{[0;s_n]}/\varphi_q^*(N_j^{[0;s_n]})$ is killed by $Q_1(Y_j)^{a_j}$ for some $a_j \geqslant 0$*

Proof. — This concerns the construction in one variable, so the proof is standard. See for example §2.2 of [**KR09**]. □

Proposition 6.2. — *The $\mathcal{R}^+(Y)$-module $\mathrm{M}^+(D)$ is stable under the Frobenius map φ_q, and $\mathrm{M}^+(D)/\varphi_q^*(\mathrm{M}^+(D))$ is killed by $Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}}$.*

Proof. — It is enough to show the proposition for $\mathrm{M}(D)^{[0;s_n]}$. Recall that

$$M_j^{[0;s_n]} = \mathcal{R}^{[0;s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}.$$

In particular, by lemma 6.1, $M_j^{[0;s_n]}$ itself is stable under φ_q. Since $M_j^{[0;s_n]}$ is stable under φ_q and $\mathrm{M}(D)^{[0;s_n]} = \cap M_j^{[0;s_n]}$, we have that $\mathrm{M}(D)^{[0;s_n]}$ is stable under φ_q.

Suppose now that $y \in \mathrm{M}(D)^{[0;s_n]}$. The fact that

$$\lambda^a \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D \subset \mathrm{M}(D)^{[0;s_n]} \subset \lambda^{-b} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D$$

implies that if $c = a + b$, then $\lambda^c y \in \varphi_q^*(\mathrm{M}(D)^{[0;s_n]})$. Let $m_1, \ldots, m_d$ be a basis of $\mathrm{M}(D)^{[0;s_n]}$ so that $\varphi_q(m_1), \ldots, \varphi_q(m_d)$ is a basis of $\varphi_q^*(\mathrm{M}(D)^{[0;s_n]})$ over $\mathcal{R}^{[0;s_n]}(Y)$, as well as of $\varphi_q^*(M_j^{[0;s_n]})$ over $\mathcal{R}^{[0;s_n]}(Y)[1/f_j]$. We can then write $\lambda^c y = \sum_i y_i \otimes m_i$ and the fact that $y \in M_j^{[0;s_n]}$ and that $M_j^{[0;s_n]}/\varphi_q^*(M_j^{[0;s_n]})$ is killed by $Q_1(Y_j)^{a_j}$ implies that y_i is divisible by $\lambda^c/Q_1(Y_j)^{a_j}$ in $\mathcal{R}^{[0;s_n]}(Y)[1/f_j]$. This shows that y_i is divisible by $\lambda^c/(Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}})$ in $\mathcal{R}^{[0;s_n]}(Y)$, which implies the second statement of the proposition. □

Remark 6.3. — Instead of working with a D where the filtrations are defined on D, we could have asked for the filtrations to be defined on $F_n \otimes_F D$ for some $n \geqslant 1$. The construction and properties of $\mathrm{M}^+(D)$ are then basically unchanged, but the determinant of φ_q is possibly more complicated than in proposition 6.2. This applies in particular to representations of G_F that become crystalline when restricted to G_{F_n} for some $n \geqslant 1$.

Definition 6.4. — A (φ_q, Γ_F)-module over $\mathcal{R}(Y)$ is a free $\mathcal{R}(Y)$-module D of finite rank, endowed with a semilinear Frobenius map φ_q, such that $\varphi_q^*(\mathrm{D}) = \mathrm{D}$, and a continuous and compatible action of Γ_F.

Theorem 6.5. — *If D is a φ_q-module with an action of Γ_F and h filtrations as above, then $\mathcal{R}(Y) \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$.*

Proof. — This follows from theorem 5.5 and proposition 6.2. □

Remark 6.6. — In [**KR09**], Kisin and Ren construct some (φ_q, Γ_F)-modules $\mathrm{M}_{\mathrm{KR}}^+(D)$ in one variable, over the ring $\mathcal{R}^+(Y_0)$, from the data of a D like ours for which the filtration $\mathrm{Fil}_j^\bullet$ is trivial for $j \neq 0$. For those D, we have $\mathrm{M}^+(D) = \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y_0)} \mathrm{M}_{\mathrm{KR}}^+(D)$. More generally, our construction shows that $\mathrm{M}^+(D)$ comes by extension of scalars from a (φ_q, Γ_F)-module in as many variables as there are nontrivial filtrations among the $\mathrm{Fil}_j^\bullet$.

Proposition 6.7. — *If $n = hk + j \geqslant h$, then the map*

$$\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\iota_n(\mathcal{R}^+(Y))} \iota_n(\mathrm{M}^+(D)) \to \mathrm{Fil}_{-j}^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^{-j}} D)$$

is an isomorphism.

Proof. — The proof is the same as that of lemma II.1.5 of [**Ber08b**]. □

Suppose now that D comes from an F-linear crystalline representation V of G_F as in example 5.1. In this case, $\mathrm{Fil}^0_j(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^j} D) = \mathbf{B}^+_{\mathrm{dR}} \otimes_F^{\varphi^j} V$. Moreover, one recovers V from D by the formula:

$$V = \{y \in (\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_F D)^{\varphi_q=1},\ y \in \mathrm{Fil}^0_j(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\varphi^j} D) \text{ for all } 0 \leqslant j \leqslant h-1\}.$$

Recall that we have constructed in §3 an injective map $\mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. This way we get a map

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_F D \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_F V.$$

Let $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ be the rings defined in §2.3 [**Ber02**]. Recall that $n(r)$ is the smallest n such that $r \leqslant p^{n-1}(p-1)$. We have the following lemma.

Lemma 6.8. — *If $y \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t]$ satisfies $\varphi^{-n}(y) \in \mathbf{B}^+_{\mathrm{dR}}$ for all $n \geqslant n(r)$, then $y \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$.*

Proof. — See lemma 1.1 of [**Ber09**] and the proof of proposition 3.2 in ibid. □

Theorem 6.9. — *If D comes from a crystalline representation V, and if $r \geqslant p^{h-1}(p-1)$, then the map above gives rise to an isomorphism*

$$\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) \to \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_F V.$$

Proof. — We first check that the image of the map above belongs to $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_F V$. If $y \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$, then its image is in $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_F V$ and satisfies $\varphi^{-n}(y) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_F^{\varphi^{-n}} V$ for all $n \geqslant n(r)$, so the assertion follows from lemma 6.8.

By proposition 6.7, the map

$$\mathbf{B}^+_{\mathrm{dR}} \otimes_{\iota_n(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})} \iota_n(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)) \to \mathbf{B}^+_{\mathrm{dR}} \otimes_F^{\varphi^{-n}} V$$

is an isomorphism if $n \geqslant n(r)$. The two $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$-modules $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ and $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_F V$ therefore have the same localizations at all $n \geqslant n(r)$, and are both stable under G_F, so that they are equal by the same argument as in the proof of lemma 2.2.2 of [**Ber08a**] (the idea is to take determinants, so that one is reduced to showing that if $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ generates an ideal stable under G_F, and has the property that $\iota_n(x)$ is a unit of $\mathbf{B}^+_{\mathrm{dR}}$ for all $n \geqslant n(r)$, then x is a unit of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$). □

7. Crystalline (φ_q, Γ_F)-modules

Let M be a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$. In this section, we define what it means for M to be crystalline, and we prove that every crystalline (φ_q, Γ_F)-module M is of the form $\mathrm{M} = \mathrm{M}(D)$, where D is a filtered φ_q-module on which the action of G_F is trivial. The results are similar to those of [**Ber08b**], which deals with the cyclotomic case.

Lemma 7.1. — *We have* $\mathrm{Frac}(\mathcal{R}(Y))^{\Gamma_F} = F$.

Proof. — If $x \in \mathrm{Frac}(\mathcal{R}(Y))^{\Gamma_F}$, then we can write $x = a/b$ with $a, b \in \mathcal{R}^{[s_n;s_n]}(Y)$ for some $n \gg 0$. By proposition 3.2, the series $a(u, \dots, \varphi^{h-1}(u))$ and $b(u, \dots, \varphi^{h-1}(u))$ converge in $\widetilde{\mathbf{B}}^{[r_n;r_n]}$. We can therefore see $\varphi^{-n}(a)$ and $\varphi^{-n}(b)$ as elements of $\mathbf{B}_{\mathrm{dR}}^+$, which satisfy $\varphi^{-n}(a)/\varphi^{-n}(b) \in \mathbf{B}_{\mathrm{dR}}^{G_F}$. The lemma now follows from the fact that $\mathbf{B}_{\mathrm{dR}}^{G_F} = F$. □

If M is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, then let $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = (\mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M})^{\Gamma_F}$.

Corollary 7.2. — *If* M *is a* (φ_q, Γ_F)*-module over* $\mathcal{R}(Y)$*, then* $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) \leqslant \mathrm{rk}(\mathrm{M})$.

Proof. — By a standard argument, lemma 7.1 implies that the map

$$\mathrm{Frac}(\mathcal{R}(Y)) \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \to \mathrm{Frac}(\mathcal{R}(Y)) \otimes_{\mathcal{R}(Y)} \mathrm{M}$$

is injective. □

Definition 7.3. — We say that a (φ_q, Γ_F)-module M over $\mathcal{R}(Y)$ is crystalline if $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = \mathrm{rk}(\mathrm{M})$.

For example, if D is a filtered φ_q-module on which the action of G_F is trivial, then $\mathrm{M}(D)$ is a crystalline (φ_q, Γ_F)-module.

Proposition 7.4. — *The category of crystalline* (φ_q, Γ_F)*-modules over* $\mathcal{R}(Y)$ *is a* $\otimes$*-category stable under subquotients and duals.*

Proof. — If M_1 and M_2 are crystalline, then $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}_1) \otimes \mathrm{D}_{\mathrm{cris}}(\mathrm{M}_2)$ injects into $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}_1 \otimes \mathrm{M}_2)$ so that $\mathrm{M}_1 \otimes \mathrm{M}_2$ is crystalline. If $0 \to \mathrm{M}_1 \to \mathrm{M} \to \mathrm{M}_2 \to 0$ is a short exact sequence of (φ_q, Γ_F)-modules over $\mathcal{R}(Y)$, then we have a short exact sequence $0 \to \mathrm{D}_{\mathrm{cris}}(\mathrm{M}_1) \to \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) \to \mathrm{D}_{\mathrm{cris}}(\mathrm{M}_2)$. If M is crystalline, then the inequalities $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}_1) \leqslant \mathrm{rk}(\mathrm{M}_1)$ and $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}_2) \leqslant \mathrm{rk}(\mathrm{M}_2)$ imply that we have equality, so that both M_1 and M_2 are cystalline. The assertion concerning duals now follows by taking exterior powers. □

Proposition 7.5. — *If* $f \in \mathcal{R}^{[s;+\infty[}(Y)$ *generates an ideal of* $\mathcal{R}^{[s;+\infty[}(Y)$ *that is stable under* Γ_F*, then* $f = u \cdot \prod_{j=0}^{h-1} \prod_{n \geqslant n(s)} (Q_n(Y_j)/p)^{a_{n,j}}$ *where* u *is a unit and* $a_{n,j} \in \mathbf{Z}_{\geqslant 0}$.

Proof. — Recall that a power series $f \in \mathcal{R}^I(Y)$ is a unit if and only if it has no zero in the corresponding domain of convergence (by the nullstellensatz, see §7.1.2 of [**BGR84**]).

Let $I = [s; u]$ be a closed subinterval of $[s; +\infty[$, so that $f \in \mathcal{R}^I(Y)$, and let $z = (z_0, z_1, \dots, z_{h-1})$ be a point such that $f(z) = 0$. Let J be the set of indices such that z_j is not a torsion point of LT_h and let $f_J \in \mathcal{R}^I_{F_k}(\{Y_j\}_{j \in J})$ be the power series that results from evaluation of the Y_m at z_m for all the z_m that are torsion points of LT_h (here k is large enough so that all those z_m belong to F_k). The ideal of $\mathcal{R}^I_{F_k}(\{Y_j\}_{j \in J})$ generated by the power series f_J is stable under $1 + p^k \mathcal{O}_F$, so that the set of its zeroes is stable under

the action of $1+p^k\mathcal{O}_F$. Furthermore, f_J has a zero none of whose coordinates are torsion points of LT_h. The same argument as in the proof of proposition 2.4 shows that $f_J=0$.

If we denote by $Z_I(f)$ the zero set of $f\in\mathcal{R}^I(Y)$, then the preceding argument shows that $Z_I(f)$ is the union of finitely many components of the form $Z_0\times\cdots Z_{h-1}$ where for each j, either Z_j is a torsion point of LT_h or $Z_j=Z_I(\{0\})$. For reasons of dimension, each of these components has precisely one Z_j which is a torsion point, the remaining $h-1$ being $Z_I(\{0\})$. This implies that in $\mathcal{R}^I(Y)$, f is the product of finitely many $Q_n(Y_j)$ by a unit.

The proposition now follows by a standard infinite factorisation argument, by writing $[s;+\infty[=\cup_{u\geqslant s}[s;u]$. □

Corollary 7.6. — *If* M *is a crystalline* (φ_q,Γ_F)*-module over* $\mathcal{R}(Y)$*, then the map*

$$\mathcal{R}(Y)[1/t]\otimes_F \mathrm{D}_{\mathrm{cris}}(\mathrm{M})\to\mathcal{R}(Y)[1/t]\otimes_{\mathcal{R}(Y)}\mathrm{M}$$

is an isomorphism.

Proof. — The map is injective by lemma 7.1, and its determinant generates an ideal of $\mathcal{R}(Y)[1/t]$ that is stable under Γ_F. Proposition 7.5 implies that this ideal is the unit ideal of $\mathcal{R}(Y)[1/t]$, and therefore that the map is an isomorphism. □

We now consider filtrations on $\mathrm{D}_{\mathrm{cris}}(\mathrm{M})$.

Lemma 7.7. — *Let D be a filtered F-vector space with a trivial action of G_F, and let W be a $\mathbf{B}_{\mathrm{dR}}^+$-lattice of $\mathbf{B}_{\mathrm{dR}}\otimes_F D$ that is stable under G_F. If we set $\mathrm{Fil}^i D=D\cap t^i\cdot W$, then $W=\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}\otimes_F D)$.*

Proof. — Let $e_1,\dots,e_d$ be a basis of D adapted to its filtration, with $e_i\in\mathrm{Fil}^{h_i}\setminus\mathrm{Fil}^{h_i+1}D$. We then have $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}\otimes_F D)=\oplus_{i=1}^d\mathbf{B}_{\mathrm{dR}}^+\cdot t^{-h_i}e_i$. By definition, we have $t^{-h_i}e_i\in W$, so that $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}\otimes_F D)\subset W$. We now prove the reverse inclusion.

If $w\in W$, then we can write $w=a_1t^{-h_1}e_1+\cdots+a_dt^{-h_d}e_d$ with $a_i\in\mathbf{B}_{\mathrm{dR}}$ and we need to prove that $a_i\in\mathbf{B}_{\mathrm{dR}}^+$ for all i. If this is not the case, then there exists $n\geqslant 1$ such that if we set $b_i=t^na_i$, then we have $b_1t^{-h_1}e_1+\cdots+b_dt^{-h_d}e_d\in t\cdot W$, with $b_i\in(\mathbf{B}_{\mathrm{dR}}^+)^\times$ for at least one i. Consider the shortest such relation; in particular, $b_i\in(\mathbf{B}_{\mathrm{dR}}^+)^\times$ for all i for which $b_i\neq 0$, and we can assume that $b_i=1$ for at least one i. If $g\in G_F$, then applying $1-\chi_{\mathrm{cycl}}(g)^{h_i}g$ to the relation yields a shorter relation, which is therefore zero. This implies that $(1-\chi_{\mathrm{cycl}}(g)^{h_i}g)(b_jt^{-h_j})=b_jt^{-h_j}$ for all $g\in G_F$ and hence that $b_j\in F\cdot t^{h_j-h_i}$. The relation above therefore reduces to an F-linear combination of those e_j for which $h_j=h_i$, belonging to $D\cap t^{h_i+1}W=\mathrm{Fil}^{h_i+1}D$, and is hence trivial. This proves that $W\subset\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}\otimes_F D)$. □

Definition 7.8. — Let M be a crystalline (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, which we can write as $\mathrm{M} = \mathcal{R}(Y) \otimes_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}$ for some s large enough. For $m \gg 0$ and $j = 0, \dots, h-1$ and $n = hm - j$, define

$$\mathrm{Fil}^i_j(\mathrm{D}_{\mathrm{cris}}(\mathrm{M})) = \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) \cap t^i \cdot (\mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}}_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}).$$

Proposition 7.9. — *The definition of* $\mathrm{Fil}^i_j(\mathrm{D}_{\mathrm{cris}}(\mathrm{M}))$ *does not depend on* $m \gg 0$*, and we have* $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes^{\varphi^{-n}}_F \mathrm{D}_{\mathrm{cris}}(\mathrm{M})) = \mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}}_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}$.

Proof. — If s is large enough, then $\mathrm{M}^{[qs;+\infty[} = \varphi_q^*(\mathrm{M}^{[s;+\infty[})$ so that

$$\mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n-h}}_{\mathcal{R}^{[qs;+\infty[}(Y)} \mathrm{M}^{[qs;+\infty[} = \mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}\varphi_q^{-1}}_{\mathcal{R}^{[qs;+\infty[}(Y)} \varphi_q(\mathrm{M}^{[s;+\infty[}) = \mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}}_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[},$$

which implies the first statement. The second statement follows from lemma 7.7, applied to $W = \mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}}_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}$. □

Theorem 7.10. — *The functors* $\mathrm{M} \mapsto \mathrm{D}_{\mathrm{cris}}(\mathrm{M})$ *and* $D \mapsto \mathrm{M}(D)$*, between the category of crystalline* (φ_q, Γ_F)*-modules over* $\mathcal{R}(Y)$ *and the category of* φ_q*-modules with* h *filtrations, are mutually inverse and give rise to an equivalence of categories.*

Proof. — If D is a φ_q-module with h filtrations, then it is clear that $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}(D)) = D$ as φ_q-modules. The fact that $\mathrm{Fil}^i_j(D) = D \cap t^i \cdot \mathrm{Fil}^0_j(\mathbf{B}_{\mathrm{dR}} \otimes^{\varphi^{-n}}_F D)$ follows from taking a basis of D adapted to $\mathrm{Fil}^\bullet_j$ and

$$\mathrm{Fil}^0_j(\mathbf{B}_{\mathrm{dR}} \otimes^{\varphi^{-n}}_F D) = \mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}}_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}(D) = \mathrm{Fil}^0_j(\mathbf{B}_{\mathrm{dR}} \otimes^{\varphi^{-n}}_F \mathrm{D}_{\mathrm{cris}}(\mathrm{M}(D)))$$

by propositions 6.7 and 7.9, so that the filtrations on D and $\mathrm{D}_{\mathrm{cris}}(\mathrm{M})$ are the same.

We now check that if M is a crystalline (φ_q, Γ_F)-module over $\mathcal{R}(Y)$ and $D = \mathrm{D}_{\mathrm{cris}}(\mathrm{M})$ with the filtration given in definition 7.8, then $\mathrm{M} = \mathrm{M}(D)$. Corollary 7.6 says that we have $\mathcal{R}(Y)[1/t] \otimes_F D = \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M}$. The theorem now follows from proposition 7.9 and the fact that if $y \in \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M}$, then $y \in \mathrm{M}$ if and only if $y \in \mathbf{B}^+_{\mathrm{dR}} \otimes^{\varphi^{-n}}_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}$ for all $n \gg 0$ by theorem 3.6. □

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